

On the trace approximations of products of Toeplitz matrices^{*}

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Abstract

The paper establishes error orders for integral limit approximations to the traces of products of Toeplitz matrices generated by integrable real symmetric functions defined on the unit circle. These approximations and the corresponding error bounds are of importance in the statistical analysis of discrete-time stationary processes: asymptotic distributions and large deviations of Toeplitz type random quadratic forms, estimation of the spectral parameters and functionals, etc.

Key words. Toeplitz matrix, Trace approximation, Error bound, Stationary process, Spectral density.

1 Introduction

Toeplitz matrices, which have great independent interest and a wide range of applications in different fields of science (economics, engineering, finance, hydrology, physics, etc.), arise naturally in the statistical analysis of stationary processes - the covariance matrix of a discrete-time stationary process is a Toeplitz matrix generated by the spectral density of that process, and vice versa, any non-negative summable function generates a Toeplitz matrix, which can be considered as a spectral density of some discrete-time stationary process, and therefore the corresponding Toeplitz matrix will be the covariance matrix of that process.

The present paper is devoted to the problem of approximation of the traces of products of Toeplitz matrices generated by integrable real symmetric functions defined on the unit circle, and estimation of the corresponding errors.

The trace approximation problem and its applications in the statistical analysis and prediction of discrete-time stationary processes go back to the classical monograph by [10]. Later this problem for different classes of generating functions (symbols) has been considered by many authors (see, e.g., [14], [12], [15], [5], [1], [6], [4], [9], [16], [13], [7], [8], and references therein). Notice that the trace approximation problem is of particular importance in the cases where the symbols of the underlying Toeplitz matrices have singularities. For instance, such cases arise in many problems of statistical analysis (asymptotic distributions and large deviations of Toeplitz type random quadratic forms, estimation of the spectral parameters and functionals, etc.) of long-memory (the spectral density is unbounded) and anti-persistent (the spectral density has zeros) discrete-time stationary processes (see, e.g., [11], [5], [1], [4], [9], [2], [16], [13], [7]), [8]).

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The paper is organized as follows. In the remainder of this section we review and summarize some known results concerning trace approximation problem. In Section 2 we state the main results of the paper and discuss two examples. Section 3 is devoted to the proofs of results stated in Section 2.

Throughout the paper the letters C , c and M , with or without index, are used to denote positive constants, the values of which can vary from line to line. Also, all functions considered in this paper are assumed to be 2π -periodic and periodically extended to \mathbb{R} .

Let $f(\lambda)$ and $g(\lambda)$ be integrable real symmetric functions defined on $\mathbb{T} := [-\pi, \pi]$, and let $T_n(f)$ and $T_n(g)$ be the $(n \times n)$ Toeplitz matrices generated by functions $f(\lambda)$ and $g(\lambda)$, respectively: for $u(\lambda) \in L^1(\mathbb{T})$ we define

$$T_n(u) = \|\widehat{u}(k-j)\|_{k,j=\overline{1,n}}, \quad n = 1, 2, \dots, \quad (1)$$

where

$$\widehat{u}(k) = \int_{\mathbb{T}} e^{i\lambda k} u(\lambda) d\lambda, \quad k \in \mathbb{Z} \quad (2)$$

are the Fourier coefficients of $u(\lambda)$.

Let ν be an arbitrary fixed positive integer. Define

$$S_{n,\nu} := S_{n,\nu}(f, g) = \frac{1}{n} \text{tr}[T_n(f)T_n(g)]^\nu, \quad (3)$$

$$M_\nu := M_\nu(f, g) = (2\pi)^{2\nu-1} \int_{-\pi}^{\pi} [f(\lambda)g(\lambda)]^\nu d\lambda \quad (4)$$

and set

$$\Delta_{n,\nu} := \Delta_{n,\nu}(f, g) = |S_{n,\nu} - M_\nu|. \quad (5)$$

The problem is to approximate $S_{n,\nu}$ by M_ν and estimate the error rate for $\Delta_{n,\nu}$. More precisely, find conditions on functions $f(\lambda)$ and $g(\lambda)$ such that:

$$(A): \quad \Delta_{n,\nu}(f, g) = o(1) \quad \text{as } n \rightarrow \infty, \quad \text{or} \quad (6)$$

$$(B): \quad \Delta_{n,\nu}(f, g) = O(n^{-\gamma}) \quad \text{for some } \gamma > 0 \quad \text{as } n \rightarrow \infty. \quad (7)$$

In Theorems A and B below we summarize some known results concerning Problems (A) and (B), respectively.

Theorem A. *Each of the following conditions is sufficient for*

$$\Delta_{n,2}(f, g) = |S_{n,2} - M_2| = o(1) \quad \text{as } n \rightarrow \infty.$$

(A1) $f(\lambda) \in L^p(\mathbb{T})$ ($p \geq 1$) and $g(\lambda) \in L^q(\mathbb{T})$ ($q \geq 1$) with $1/p + 1/q \leq 1/2$.

(A2) $f \in L^2(\mathbb{T})$, $g \in L^2(\mathbb{T})$, $fg \in L^2(\mathbb{T})$ and

$$\int_{\mathbb{T}} f^2(\lambda)g^2(\lambda - \mu) d\lambda \longrightarrow \int_{\mathbb{T}} f^2(\lambda)g^2(\lambda) d\lambda \quad \text{as } \mu \rightarrow 0. \quad (8)$$

(A3) *The function*

$$\varphi(\mathbf{u}) = \varphi(u_1, u_2, u_3) = \int_{\mathbb{T}} f(\lambda)g(\lambda - u_1)f(\lambda - u_2)g(\lambda - u_3) d\lambda, \quad (9)$$

belongs to $L^2(\mathbb{T}^3)$ and is continuous at $\mathbf{0} = (0, 0, 0)$.

(A4) $f(\lambda) \leq |\lambda|^{-\alpha} L_1(\lambda)$ and $|g(\lambda)| \leq |\lambda|^{-\beta} L_2(\lambda)$ for $\lambda \in \mathbb{T}$ and some $\alpha < 1$, $\beta < 1$, $\alpha + \beta \leq 1/2$, and $L_i \in SV(\mathbb{R})$, $\lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2(\mathbb{T})$, $i = 1, 2$, where $SV(\mathbb{R})$ is the class of slowly varying at zero functions $u(\lambda)$, $\lambda \in \mathbb{R}$ satisfying $u(\lambda) \in L^\infty(\mathbb{R})$, $\lim_{\lambda \rightarrow 0} u(\lambda) = 0$, $u(\lambda) = u(-\lambda)$ and $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu$.

Remark 1. Assertion (A1) was proved by [1]. For special case $p = q = \infty$, it was first established by [10], while the case $p = 2$, $q = \infty$ was proved by [14] and [12]. Assertion (A2) was proved in [9] (see, also, [8]). Assertions (A3) and (A4) were established in [7]. A special case of (A4), when $\alpha + \beta < 1/2$, was considered by [5].

Theorem B. *The following assertions hold:*

(B1) *If the Fourier coefficients $\hat{f}(k)$ and $\hat{g}(k)$ of functions $f(\lambda)$ and $g(\lambda)$ satisfy the conditions*

$$\sum_{k=-\infty}^{\infty} |k| |\hat{f}(k)| < \infty \quad \text{and} \quad \sum_{k=-\infty}^{\infty} |k| |\hat{g}(k)| < \infty, \quad (10)$$

then for $\nu = 1, 2, \dots$

$$\Delta_{n,\nu}(f, g) = O(n^{-1}) \quad \text{as } n \rightarrow \infty. \quad (11)$$

(B2) *If there exist constants C_i with $0 < C_i < \infty$, $i = 1, 2, 3, 4$, such that*

$$\sup_{\lambda \in [-\pi, \pi]} |f(\lambda)| \leq C_1, \quad \sup_{\lambda \in [-\pi, \pi]} |g(\lambda)| \leq C_2, \quad (12)$$

$$\sup_{\lambda \in [-\pi, \pi]} |f'(\lambda)| \leq C_3, \quad \sup_{\lambda \in [-\pi, \pi]} |g'(\lambda)| \leq C_4, \quad (13)$$

then for any $\epsilon > 0$ and $\nu = 1, 2, \dots$

$$\Delta_{n,\nu}(f, g) = O(n^{-1+\epsilon}) \quad \text{as } n \rightarrow \infty. \quad (14)$$

Remark 2. Assertion (B1) was established in [15] (see, also, [16]). Assertions (B2) was proved in [13]. Note that in (B2) the asymptotic relation (14) is valid under the single condition (13) because (13) obviously implies (12).

Remark 3. In [13] was also stated the following result (see [13], Theorem 2).

(B3) *Assume that the functions $f(\lambda)$ and $g(\lambda)$ satisfy the conditions:*

(a) *$f(\lambda)$ and $g(\lambda)$ are symmetric, real valued, continuously differentiable at all $\lambda \neq 0$ and there exist $0 < C_i < \infty$, $i = 1, 2$, such that for any $\lambda \in [-\pi, \pi]$*

$$|f(\lambda)| \leq C_1 |\lambda|^{-\alpha}, \quad |g(\lambda)| \leq C_2 |\lambda|^{-\beta}, \quad \alpha < 1, \beta < 1.$$

(b) *For all $t > 0$ there exist M_{t1} and M_{t2} such that*

$$\sup_{|\lambda| > t} |f'(\lambda)| \leq M_{t1} \quad \text{and} \quad \sup_{|\lambda| > t} |g'(\lambda)| \leq M_{t2}.$$

(c) *$\nu(\alpha + \beta) < 1$, $\nu \in \mathbb{N}$.*

Then for any $\epsilon > 0$

$$\Delta_{n,\nu}(f, g) = \begin{cases} O(n^{-1+\nu(\alpha+\beta)+\epsilon}), & \text{if } \alpha + \beta > 0 \\ O(n^{-1+\epsilon}), & \text{if } \alpha + \beta \leq 0. \end{cases} \quad (15)$$

First observe that condition (a) in (B3) implies condition (b).

Unfortunately, the proof of (B3) given in [13] contains an inaccuracy. The issue is that the authors assertion that "*the last integral in formula (26) is finite under the conditions (27)*" ([13], p. 743), is not correct.

More precisely, they state that for some $t \in (0, \pi)$ the integral

$$I := \int_{A_t} |z_1|^{2\nu\eta-\nu(\alpha+\beta)-1} |z_2 \cdots z_{2\nu} - 1|^{\eta-1} \\ \times |z_2 - 1|^{\eta-1} \cdots |z_{2\nu} - 1|^{\eta-1} dz_1 \cdots dz_{2\nu}, \quad (16)$$

where

$$A_t := \{(z_1, \dots, z_{2\nu}) \in \mathbb{R}^{2\nu} : |z_1| \leq t, |z_1 z_2| \leq t, \dots, |z_1 \cdots z_{2\nu}| \leq t, \\ |z_1| > \frac{1}{2}|z_1 z_2| > \cdots > \frac{1}{2^{2\nu-1}}|z_1 \cdots z_{2\nu}| > \frac{1}{2^{2\nu}}|z_1|\} \quad (17)$$

converges (is finite) in the parameter set (see (c) and [13], formulas (26) and (27)):

$$\frac{1}{2}(\alpha + \beta) < \eta < 1, \quad 0 < \alpha + \beta < \frac{1}{\nu}, \quad (18)$$

and then conclude that the quantity

$$J_n := \frac{C}{n^{1-2\nu\eta}} \times I, \quad (19)$$

where C is a constant, goes to zero as $n \rightarrow \infty$ with the specified rate.

First observe that to have $J_n \rightarrow 0$ as $n \rightarrow \infty$, the condition $1 - 2\nu\eta > 0$ should be imposed, that is, along with (18), the parameter η should also satisfy

$$\eta < \frac{1}{2\nu}. \quad (20)$$

The arguments that follow, show that the integral in (16) diverges in the parameter set (18), (20).

We first prove the following inequality: for $0 < \gamma < 1$, $0 < \theta < 1$ and $y_0 > 2$

$$\int_1^2 (xy - 1)^{-\theta} (x - 1)^{-\gamma} dx \geq c (y - 1)^{1-\gamma-\theta}, \quad 1 < y < y_0, \quad (21)$$

where the constant c depends only on γ , θ and y_0 . To prove (21), observe first that for $1 \leq x \leq y \leq 2$,

$$xy - 1 = (x - 1)(y - 1) + (x - 1) + (y - 1) \leq 3(y - 1).$$

Consequently,

$$\int_1^2 (xy - 1)^{-\theta} (x - 1)^{-\gamma} dx \geq \int_1^y (xy - 1)^{-\theta} (x - 1)^{-\gamma} dx \geq \\ \geq 3^{-\theta} (y - 1)^{-\theta} \int_1^y (x - 1)^{-\gamma} dx = \frac{3^{-\theta}}{(1 - \gamma)} \cdot (y - 1)^{1-\gamma-\theta},$$

yielding (21) for $1 < y \leq 2$.

For $2 < y < y_0$ we have $(y-1)^{1-\gamma-\theta} < y_0$ and

$$\begin{aligned} \int_1^2 (xy-1)^{-\theta} (x-1)^{-\gamma} dx &\geq \int_1^2 (xy_0-1)^{-\theta} (x-1)^{-\gamma} dx \\ &=: J \geq \frac{J}{y_0} \cdot (y-1)^{1-\gamma-\theta}, \end{aligned}$$

where J depends only on γ, θ and y_0 . Inequality (21) is proved.

Now, setting $\varepsilon := 2\nu\eta - \nu(\alpha + \beta) > 0$, taking into account that (see (17))

$$A_t \supset \left\{ (z_1, \dots, z_{2\nu}) \in \mathbb{R}^{2\nu} : 0 < z_1 < \frac{t}{2\nu}, 1 < z_i < 2, i = 2, 3, \dots, 2\nu \right\},$$

and applying (21) with $y_0 = 2^{2\nu}$ successively $(2\nu - 2)$ times, we get

$$\begin{aligned} I &= \int_{A_t} |z_1|^{\varepsilon-1} |z_2 \cdots z_{2\nu} - 1|^{\eta-1} |z_2 - 1|^{\eta-1} \cdots |z_{2\nu} - 1|^{\eta-1} dz_1 \cdots dz_{2\nu} \\ &\geq \int_0^{t/2\nu} z_1^{\varepsilon-1} dz_1 \int_1^2 \cdots \int_1^2 (z_2 z_3 \cdots z_{2\nu} - 1)^{\eta-1} (z_2 - 1)^{\eta-1} dz_2 \\ &\quad \times (z_3 - 1)^{\eta-1} \cdots (z_{2\nu} - 1)^{\eta-1} dz_3 \cdots dz_{2\nu} \\ &\geq c \int_0^{t/2\nu} z_1^{\varepsilon-1} dz_1 \int_1^2 \cdots \int_1^2 (z_3 \cdots z_{2\nu} - 1)^{2\eta-1} (z_3 - 1)^{\eta-1} dz_3 \\ &\quad \times (z_4 - 1)^{\eta-1} \cdots (z_{2\nu} - 1)^{\eta-1} dz_4 \cdots dz_{2\nu} \\ &\geq \dots\dots\dots \\ &\geq c_1 \int_0^{t/2\nu} z_1^{\varepsilon-1} dz_1 \int_1^2 (z_{2\nu} - 1)^{2\nu\eta-2} dz_{2\nu}. \end{aligned}$$

The last integral diverges, since by (20) $2\nu\eta - 2 < -1$.

In this paper we prove the asymptotic relation

$$\Delta_{n,\nu}(f, g) = O(n^{-\gamma}), \quad \gamma > 0, \quad \text{as } n \rightarrow \infty \quad (22)$$

for some classes of generating functions $f(\lambda)$ and $g(\lambda)$. The results improve some of the $o(1)$ rates stated in Theorem A. For simplicity we state and prove the results in the typical special case where $\nu = 2$.

2 Error bounds for $\Delta_{n,2}$

For $\psi \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$ we denote by $\omega_p(\psi, \delta)$ the L^p -modulus of continuity of ψ :

$$\omega_p(\psi, \delta) := \sup_{0 < h \leq \delta} \|\psi(\cdot + h) - \psi(\cdot)\|_p, \quad \delta > 0.$$

Given numbers $0 < \gamma \leq 1$ and $1 \leq p \leq \infty$, we denote by $\text{Lip}(p, \gamma) = \text{Lip}(\mathbb{T}; p, \gamma)$ the L^p -Lipschitz class of functions defined on \mathbb{T} (see, e.g., [3]):

$$\text{Lip}(p, \gamma) = \{\psi(\lambda) \in L^p(\mathbb{T}); \quad \omega_p(\psi; \delta) = O(\delta^\gamma), \quad \delta \rightarrow 0\}.$$

Observe that if $\psi \in \text{Lip}(p, \gamma)$, then there exists a constant C such that $\omega_p(\psi; \delta) \leq C \delta^\gamma$ for all $\delta > 0$.

The main results of the paper are the following theorems.

Theorem 1. *Let the function $\varphi(\mathbf{u}) = \varphi(u_1, u_2, u_3)$ be as in (9). Assume that with some constants $C > 0$ and $\gamma \in (0, 1]$*

$$|\varphi(\mathbf{u}) - \varphi(\mathbf{0})| \leq C|\mathbf{u}|^\gamma, \quad \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3, \quad (23)$$

where $\mathbf{0} = (0, 0, 0)$ and $|\mathbf{u}| = |u_1| + |u_2| + |u_3|$. Then for any $\varepsilon > 0$

$$\Delta_{n,2}(f, g) = O(n^{-\gamma+\varepsilon}) \quad \text{as } n \rightarrow \infty. \quad (24)$$

The next two theorems we will deduce from Theorem 1, and hence can be considered as corollaries of Theorem 1.

Theorem 2. *Let $f \in \text{Lip}(p, \gamma)$ and $g \in \text{Lip}(q, \gamma)$ with $\gamma \in (0, 1]$ and $p, q \geq 1$ such that $1/p + 1/q \leq 1/2$. Then (24) holds for any $\varepsilon > 0$.*

Theorem 3. *Let $f_i(\lambda)$, $i = 1, 2$, be two differentiable functions on $[-\pi, \pi] \setminus \{0\}$, such that for some constants $\alpha_i > 0$, $i = 1, 2$, satisfying $\alpha_1 + \alpha_2 < 1/2$ and $M_{1i}, M_{2i} > 0$, $i = 1, 2$*

$$|f_i(\lambda)| \leq M_{1i}|\lambda|^{-\alpha_i}, \quad |f'_i(\lambda)| \leq M_{2i}|\lambda|^{-(\alpha_i+1)}, \quad \lambda \in [-\pi, \pi] \setminus \{0\}. \quad (25)$$

Then for any $\varepsilon > 0$

$$\Delta_{n,2}(f_1, f_2) = O(n^{-\gamma+\varepsilon}) \quad \text{as } n \rightarrow \infty \quad (26)$$

with

$$\gamma = \frac{1}{4} - \frac{\alpha_1 + \alpha_2}{2}. \quad (27)$$

Example 1. Let $f_i(\lambda) = |\lambda|^{-\alpha_i}$, $\lambda \in [-\pi, \pi]$, $i = 1, 2$, with $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 < 1/2$. It is easy to see that the conditions of Theorem 3 are satisfied, and hence we have (26) with γ as in (27).

Example 2. Let $f_i(\lambda)$, $\lambda \in [-\pi, \pi]$, $i = 1, 2$, be the spectral density functions of two long-memory discrete-time stationary processes given by

$$f_i(\lambda) = \frac{\sigma_i^2}{2\pi} |1 - e^{i\lambda}|^{-\alpha_i} \quad (28)$$

with $0 < \sigma_i^2 < \infty$, $\alpha_i > 0$, $i = 1, 2$, and $\alpha_1 + \alpha_2 < 1/2$. Then (26) holds with γ as in (27).

Indeed, assuming that $\lambda \in (0, \pi]$ (the case $\lambda \in [-\pi, 0)$ is treated similarly), and taking into account $|1 - e^{i\lambda}| = 2 \sin(\lambda/2)$, we have for $i = 1, 2$

$$f_i(\lambda) = \frac{\sigma_i^2}{2\pi} \cdot 2^{-\alpha_i} \left[\sin \frac{\lambda}{2} \right]^{-\alpha_i}$$

and

$$f'_i(\lambda) = \frac{\sigma_i^2}{2\pi} \cdot \left[-\alpha_i 2^{-\alpha_i-1} \left(\sin \frac{\lambda}{2} \right)^{-\alpha_i-1} \cos \frac{\lambda}{2} \right].$$

It is clear that the conditions of Theorem 3 are satisfied with $M_{1i} = M_{2i} = \sigma_i^2$, $i = 1, 2$, and the result follows.

Remark 4. It is easy to see that under the conditions of Theorem (B2) we have $f \in \text{Lip}(p, 1)$ and $g \in \text{Lip}(p, 1)$ for any $p \geq 1$. Hence Theorem 2 implies Theorem (B2) (for $\nu = 2$).

Remark 5. For functions $f_i(\lambda)$, $i = 1, 2$, defined by (28) an explicit second-order expansion for $S_{n,1}$ (see (3)) was found by [13], where they showed that in this special case the second-order expansion removes the singularity in the first-order approximation, and provides an improved approximation of order $\gamma = 1 - 2(\alpha_1 + \alpha_2)$.

3 Proofs

We first state a number of lemmas. The results of the first two lemmas are known (see, e.g., [8], p. 8, 161).

Lemma 1. *Let $D_n(u)$ be the Dirichlet kernel*

$$D_n(u) = \frac{\sin(nu/2)}{\sin(u/2)}. \quad (29)$$

Then, for any $\delta \in [0, 1]$ and $u \in \mathbb{T}$

$$|D_n(u)| \leq \pi n^\delta |u|^{\delta-1}. \quad (30)$$

Lemma 2. *Let $0 < \beta < 1$, $0 < \alpha < 1$, and $\alpha + \beta > 1$. Then for any $y \in \mathbb{R}$, $y \neq 0$,*

$$\int_{\mathbb{R}} \frac{1}{|x|^\alpha |x+y|^\beta} dx = \frac{M}{|y|^{\alpha+\beta-1}}, \quad (31)$$

where M is a constant depending on α and β .

Lemma 3. *Let $0 < \alpha \leq 1$ and $\frac{2}{3} < \beta < \frac{\alpha+3}{4}$. Then*

$$B_i := \int_{\mathbb{T}^3} \frac{|u_i|^\alpha}{|u_1 u_2 u_3 (u_1 + u_2 + u_3)|^\beta} du_1 du_2 du_3 < \infty, \quad i = 1, 2, 3. \quad (32)$$

Proof. Using Lemma 2 we can write

$$\begin{aligned} B_1 &\leq \int_{|u_1| \leq \pi} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbb{R}} \frac{1}{|u_2|^\beta} \int_{\mathbb{R}} \frac{1}{|u_3(u_1 + u_2 + u_3)|^\beta} du_3 du_2 du_1 \\ &= M \int_{|u_1| \leq \pi} \frac{1}{|u_1|^{\beta-\alpha}} \int_{\mathbb{R}} \frac{1}{|u_2|^\beta |u_1 + u_2|^{2\beta-1}} du_2 du_1 \\ &= M^2 \int_{|u_1| \leq \pi} \frac{1}{|u_1|^{4\beta-\alpha-2}} du_1 < \infty, \end{aligned}$$

yielding (32) for $i = 1$. The quantities B_2 and B_3 can be estimated in the same way. \square

Lemma 4. *Let $p > 1$ and $0 < \alpha < 1$ be such that $\alpha p < 1$, and for some constants $M_1, M_2 > 0$*

$$|f(\lambda)| \leq M_1 |\lambda|^{-\alpha}, \quad |f'(\lambda)| \leq M_2 |\lambda|^{-(\alpha+1)}, \quad \lambda \in [-\pi, \pi], \quad \lambda \neq 0. \quad (33)$$

Then $f \in \text{Lip}(p, 1/p - \alpha)$.

Proof. Let $h \in (0, 1)$ be fixed. Then

$$\int_{|\lambda| \leq 2h} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq (2M_1)^p \int_0^{3h} \lambda^{-p\alpha} d\lambda \leq Ch^{1-p\alpha}. \quad (34)$$

Next, for $|\lambda| > 2h$ we have with some $\xi \in (\lambda, \lambda + h)$

$$|f(\lambda + h) - f(\lambda)| = |f'(\xi) \cdot h| \leq M_2 h |\xi|^{-(\alpha+1)}.$$

Hence

$$\int_{2h < |\lambda| < \pi} |f(\lambda + h) - f(\lambda)|^p d\lambda \leq Ch^p \int_h^\pi \lambda^{-p(\alpha+1)} d\lambda \leq Ch^{1-p\alpha}. \quad (35)$$

From (34) and (35) we get

$$\|f(\lambda + h) - f(\lambda)\|_p \leq Ch^{1/p-\alpha},$$

showing that $f \in Lip(p, 1/p - \alpha)$. \square

Proof of Theorem 1. Denote

$$\Phi_n(\mathbf{u}) := \Phi_n(u_1, u_2, u_3) = \frac{1}{8\pi^3 n} \cdot D_n(u_1) D_n(u_2) D_n(u_3) D_n(u_1 + u_2 + u_3) \quad (36)$$

and

$$\Psi(\mathbf{u}) := \Psi(u_1, u_2, u_3) = \varphi(u_1, u_1 + u_2, u_1 + u_2 + u_3), \quad (37)$$

where $D_n(u)$ and $\varphi(u_1, u_2, u_3)$ are defined by (29) and (9), respectively. Then ([7])

$$\begin{aligned} \Delta_{n,2} &= \left| \frac{1}{n} \text{tr}[B_n(f)B_n(g)]^2 - 8\pi^3 \int_{\mathbb{T}} f^2(\lambda) g^2(\lambda) d\lambda \right| \\ &= \left| \int_{\mathbb{T}^3} [\Psi(\mathbf{u}) - \Psi(\mathbf{0})] \Phi_n(\mathbf{u}) d\mathbf{u} \right|, \quad \mathbf{0} = (0, 0, 0). \end{aligned} \quad (38)$$

It follows from (23) and (37) that

$$|\Psi(\mathbf{u}) - \Psi(\mathbf{0})| \leq 3C|u_1|^\gamma + 2C|u_2|^\gamma + C|u_3|^\gamma, \quad \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{T}^3. \quad (39)$$

Let $\varepsilon \in (0, \gamma)$. Then, applying Lemma 1 with $\delta = \frac{1+\varepsilon-\gamma}{4}$, and using (38) and (39), we have

$$\begin{aligned} |\Delta_{n,2}| &\leq \int_{\mathbb{T}^3} |\Psi(\mathbf{u}) - \Psi(\mathbf{0})| |\Phi_n(\mathbf{u})| d\mathbf{u} \\ &\leq \frac{2}{n^{1-4\delta}} \sum_{i=1}^3 C_i \int_{\mathbb{T}^3} \frac{|u_i|^\gamma}{|u_1 u_2 u_3 (u_1 + u_2 + u_3)|^{1-\delta}} du_1 du_2 du_3. \end{aligned}$$

This, combined with Lemma 3 implies the statement of Theorem 1. \square

Proof of Theorem 2. According to Theorem 1 it is enough to prove that the function

$$\varphi(\mathbf{t}) := \int_{\mathbb{T}} h_0(u) h_1(u - t_1) h_2(u - t_2) h_3(u - t_3) du, \quad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3 \quad (40)$$

with some positive constant C satisfies the condition

$$|\varphi(\mathbf{t}) - \varphi(\mathbf{0})| \leq C|\mathbf{t}|^\gamma, \quad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3, \quad (41)$$

provided that

$$h_i \in \text{Lip}(p_i, \gamma), \quad 1 \leq p_i \leq \infty, \quad i = 0, 1, 2, 3, \quad \text{and} \quad \sum_{i=0}^3 \frac{1}{p_i} \leq 1. \quad (42)$$

To prove (41) we fix $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3$ and denote

$$\bar{h}_i(u) = h_i(u - t_i) - h_i(u), \quad i = 1, 2, 3. \quad (43)$$

Since $h_i \in \text{Lip}(p_i, \gamma)$ we have

$$\|\bar{h}_i\|_{p_i} \leq C_i |\mathbf{t}|^\gamma, \quad i = 1, 2, 3. \quad (44)$$

By (40) and (43)

$$\varphi(\mathbf{t}) = \int_{\mathbb{T}} h_0(u) \prod_{i=1}^3 (\bar{h}_i(u) + h_i(u)) du = \varphi(\mathbf{0}) + W.$$

Each of the seven integrals comprising W contains at least one function \bar{h}_i , and in view of (44), can be estimated as follows

$$\left| \int_{\mathbb{T}} h_0(u) \bar{h}_1(u) h_2(u) h_3(u) du \right| \leq \|h_0\|_{L^{p_0}} \|\bar{h}_1\|_{L^{p_1}} \|h_2\|_{L^{p_2}} \|h_3\|_{L^{p_3}} \leq C |\mathbf{t}|^\gamma.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. For given $\alpha_i > 0$, $i = 1, 2$, satisfying $\alpha_1 + \alpha_2 < 1/2$ we set

$$\frac{1}{p_1} = \frac{1}{4} + \frac{\alpha_1 - \alpha_2}{2} \quad \text{and} \quad \frac{1}{p_2} = \frac{1}{4} + \frac{\alpha_2 - \alpha_1}{2}.$$

It is easy to check that such defined p_1 and p_2 satisfy the conditions

$$p_1 > 1, \quad p_2 > 1, \quad \alpha_1 p_1 < 1, \quad \alpha_2 p_2 < 1, \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}.$$

Hence, according to Lemma 4, $f_i \in \text{Lip}(p_i, \gamma)$, $i = 1, 2$, with

$$\gamma = \frac{1}{p_1} - \alpha_1 = \frac{1}{p_2} - \alpha_2 = \frac{1}{4} - \frac{\alpha_1 + \alpha_2}{2}.$$

Applying Theorem 2 with $p = p_1$, $q = p_2$, $f = f_1$ and $g = f_2$ we get (26) with γ as in (27). □

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